

ON PARABOLIC EQUATIONS IN ONE SPACE DIMENSION

N.V. KRYLOV

ABSTRACT. Several negative results are presented concerning the solvability in Sobolev classes of the Cauchy problem for the inhomogeneous second-order uniformly parabolic equations without lower order terms in one space dimension. The main coefficient is assumed to be a bounded measurable function of (t, x) bounded away from zero. We also discuss upper and lower estimates of certain kind on the fundamental solutions of such equations.

1. INTRODUCTION

We are going to consider functions $u(t, x)$ of two variables $t, x \in \mathbb{R}$. Denote $D = \partial/\partial x$. When it makes sense we write

$$u_t = \partial_t u = \frac{\partial u}{\partial t}, \quad u_x = Du = \frac{\partial u}{\partial x}, \quad u_{xx} = D^2 u = \frac{\partial^2 u}{(\partial x)^2}, \dots$$

Let $p \in (1, \infty)$ and denote by $W_p^{1,2}$ the set of functions u given on \mathbb{R}^2 such that $u, u_x, u_{xx}, u_t \in L_p(\mathbb{R}^2)$. The norm in $W_p^{1,2}$ is introduced in the usual way. Set

$$Q = (0, 1) \times \mathbb{R}$$

and define $\overset{0}{W}_p^{1,2}(Q)$ as the set of restrictions on Q of functions in $W_p^{1,2}$ each of which is identically zero for $t \leq 0$.

For $0 < \beta \leq \alpha < \infty$ denote by $\mathfrak{A}(\beta, \alpha)$ the set of Borel functions $a(t, x)$ on Q such that $\beta \leq a(t, x) \leq \alpha$ for all $(t, x) \in Q$. One of our main objects of investigation is the equation

$$u_t = au_{xx} + f \tag{1.1}$$

in Q . For given $a \in \mathfrak{A}(\beta, \alpha)$ and $f \in L_p(Q)$ we will look for solutions of this equation in class $W_p^{1,2}(Q)$ satisfying zero initial condition or, in other terms, for solutions in $\overset{0}{W}_p^{1,2}(Q)$.

1991 *Mathematics Subject Classification.* 35K10, 35K15.

Key words and phrases. Absence of a priori estimates, Sobolev classes, estimates on the fundamental solutions.

The work was partially supported by NSF Grant DMS-1160569.

For the author the main source of interest in the solvability question of such simplest one-dimensional equations was the theory of multidimensional parabolic equations with coefficients which are only measurable in time and one spacial variable and, say, just independent of all other variables. It turns out that if we knew that (1.1) is solvable in all $W_p^{1,2}(Q)$, then the multidimensional version of this result would be also available and would lead to much more general results about equations with partially regular coefficients and with easier proofs than those, for instance, in [2] and the reference therein. The careful reader can see it by following and sometimes slightly changing the arguments in those references.

However, it turned out that not for all $p \in (1, \infty)$ and measurable a one can guarantee the solvability of (1.1). We show here that at least for $p \notin [3/2, 3]$ there are equations which are not solvable.

This situation is quite different from what is known for two-dimensional *elliptic* equations with measurable coefficients. The fact that generally they are not solvable if p is not close to 2 is well-known and was first demonstrated by N.N. Uraltseva in 1967 (see [9] or [8]). Many more examples of impossibility of solving elliptic equations in divergence and non divergence forms in two space dimensions can be found in [1]. In our parabolic case we cannot exclude even a part of the range of $p \in [3/2, 3]$, and the author has no idea what is going on in this range.

We also provide similar results for divergence type equations.

The third line of our investigation is constructing estimates from below and from above for solutions of the Cauchy problem with $f = 0$ and the initial data that is the indicator of an interval. In the case of the estimates from above we are able to present essentially sharp estimate for a in classes $\mathfrak{A}(\beta, \alpha)$ in the full range of $0 < \beta < \alpha < \infty$. In the case of estimates from below we were only able to cover the case that $1 \geq \beta/\alpha > c$, where c is a certain number, $c > 0$. One can probably go further down to zero by considering solutions of the Ornstein-Uhlenbeck equation (3.1) for $\lambda > 1$ when solutions given in an integral form can be found in [11] or [7]. We leave trying to do this to the interested reader only conjecturing that the left inequality in (2.6) holds for any $\gamma > 1$ if $a_{1\gamma} > 0$ is chosen sufficiently small.

Pathological behavior of fundamental solutions of the Cauchy problem for parabolic equations even in one space dimension with continuous coefficient a was noticed quite a while ago in [5], where the fundamental solution of the Cauchy problem blows up at a point, say $(0, 0)$, for any $t > 0$. Then in [4] and [12] independently examples were

constructed again with continuous a in which for any $t > 0$ the fundamental solution (as a generalized function, in fact a measure) was just singular with respect to Lebesgue measure. We add to this line of research a new information about the integrals of fundamental solutions over intervals.

The article is organized as follows. In Section 2 we present our main results. Section 3 contains general results about the Ornstein-Uhlenbeck equation (3.1) when λ is arbitrary. In Section 4 we restrict our attention to $\lambda < 0$ and then use the obtained results in Section 5 to construct an essentially sharp barrier from above for the solutions of the Cauchy problem with $f = 0$ and the indicator function of an interval as the initial data. This barrier serves in Section 6 as a solution of (1.1) with $f = 0$ in class $\overset{0}{W}_p^{1,2}(Q)$ for $p \in (1, 3/2)$ and this and a duality argument ruin the hope to build a solvability theory in $W_p^{1,2}$ for non divergence type equations and $p \in (1, 3/2) \cup (3, \infty)$.

In Section 7 we deal with divergence type equations and basically use the same barrier and the observation that the x -derivative of a solution of (1.1) is a solution of a divergence type equation. The final Section 8 contains the estimate from below alluded to above.

2. MAIN RESULTS

The reader understands that equations with a of class $\mathfrak{A}(\beta, \alpha)$ can be easily transformed into equations with a of class $\mathfrak{A}(1, \alpha/\beta)$ or $\mathfrak{A}(\beta/\alpha, 1)$ by using dilations or contractions of the t -axis. Therefore, we only consider these two classes of a .

Theorem 2.1. *Let $p \in (1, 3/2)$, then there exists an $\alpha = \alpha(p) \in (1, \infty)$ and a function $a \in \mathfrak{A}(1, \alpha)$, such that equation (1.1) with $f \equiv 0$ has a nonzero unbounded solution in class $\overset{0}{W}_p^{1,2}(Q)$. Furthermore, $\alpha(p) \rightarrow 1$ as $p \downarrow 1$ and $\alpha(p) \rightarrow \infty$ as $p \uparrow 3/2$.*

Remark 2.2. This theorem shows that no matter how small the discontinuities of a are allowed, there is an a and $p > 1$ perhaps very close to 1 such that the first assertion of the theorem holds.

This theorem also implies that there is no $p \in (1, 3/2)$ such that the estimate

$$\sup_Q |u| \leq N \|u_t - au_{xx}\|_{L_p(Q)}. \quad (2.1)$$

holds for any given $a \in \mathfrak{A}(1, \alpha(p))$ with a constant, perhaps depending on a but independent of $u \in \overset{0}{W}_p^{1,2}(Q)$. Recall that according to the

parabolic Alexandrov estimate, for any $\alpha \in (1, \infty)$ there is a constant N such that (2.1) with $p = 2$ holds for all $a \in \mathfrak{A}(1, \alpha)$ and $u \in \overset{0}{W}_2^{1,2}(Q)$.

The author does not know what could be the least value of p for (2.1) to hold for any $\alpha \in (1, \infty)$, $a \in \mathfrak{A}(1, \alpha)$, and $u \in \overset{0}{W}_p^{1,2}(Q)$ with N depending only on α .

Theorem 2.3. *Let $p \in (3, \infty)$, then for $\alpha = \alpha(p/(p-1))$ there exists a function $a \in \mathfrak{A}(1, \alpha)$, such that equation (1.1) for some $f \in L_p(Q)$ does not have solutions in class $\overset{0}{W}_p^{1,2}(Q)$.*

Corollary 2.4. *For any $p \in (1, 3/2) \cup (3, \infty)$, there exists an $\alpha > 1$ and a function $a \in \mathfrak{A}(1, \alpha)$ such that, for any $N \in (0, \infty)$, the estimate*

$$\|u_{xx}\|_{L_p(Q)} \leq N \|u_t - (\lambda a + 1 - \lambda)u_{xx}\|_{L_p(Q)} \quad (2.2)$$

fails to hold for all $u \in \overset{0}{W}_p^{1,2}(Q)$ and $\lambda \in [0, 1]$, that is fails to hold on the set $\overset{0}{W}_p^{1,2}(Q) \times [0, 1]$.

Indeed, otherwise the method of continuity would prove the unique solvability of (1.1) in class $\overset{0}{W}_p^{1,2}(Q)$ for any $f \in L_p(Q)$.

The following few results relate to the divergence type equations. Set $\Lambda = (1 - D^2)^{1/2}$, $H_p^n(\mathbb{R}) = \Lambda^{-n}L_p(\mathbb{R})$, and $\overset{0}{\mathcal{H}}_p^1(Q) = \Lambda \overset{0}{W}_p^{1,2}(Q)$. For $f \in L_p(Q)$ consider the equation

$$u_t = (au_x)_x + \Lambda f \quad (2.3)$$

in Q . Solutions of this equation will be looked for in $\overset{0}{\mathcal{H}}_p^1(Q)$. Since $u = \Lambda w$ for a $w \in \overset{0}{W}_p^{1,2}(Q)$ the function u_t is $H_p^{-1}(\mathbb{R})$ -valued and so are $(au_x)_x$ and Λf . Hence, equation (1.1) has perfect sense, and u_t is the strong derivative with respect to t of u as an $H_p^{-1}(\mathbb{R})$ -valued function.

Theorem 2.5. *Let $p \in (1, 3/2)$ and $\alpha = \alpha(p)$. Then there exists a function $a \in \mathfrak{A}(1, \alpha)$, such that equation (2.3) with $f \equiv 0$ has a nonzero solution in class $\overset{0}{\mathcal{H}}_p^1(Q)$. Moreover, for this solution the functions u_t, u_x , and au_x are continuously differentiable functions of (t, x) , so that equation (2.3) holds in the classical sense everywhere in Q .*

Theorem 2.6. *Let $p \in (3, \infty)$ and $\alpha = \alpha(p)$. Then there exists a function $a \in \mathfrak{A}(1, \alpha)$, such that equation (2.3) for some $f \in L_p(Q)$ does not have solutions in class $\overset{0}{\mathcal{H}}_p^{1,2}(Q)$.*

Similarly to Corollary 2.4 we have the following.

Corollary 2.7. *For any $p \in (1, 3/2) \cup (3, \infty)$, there exists an $\alpha > 1$ and a function $a \in \mathfrak{A}(1, \alpha)$ such that, for any $N \in (0, \infty)$, the estimate*

$$\|u_x\|_{L_p(Q)} \leq N\|f\|_{L_p(Q)}$$

fails to hold on the set of all couples (λ, u) , where $\lambda \in [0, 1]$ and $u \in \mathring{\mathcal{H}}_p^1(Q)$ is a solution of

$$u_t = ([\lambda a + 1 - \lambda]u_x)_x + \Lambda f$$

with $f \in L_p(Q)$.

Remark 2.8. The author does not know what is going on concerning the above results for the whole region of values of $p \in [3/2, 3]$.

In spite of the above negative results, for any $\alpha > 1$, if p is sufficiently close to 2, there is a constant N such that (2.2) holds for all $u \in \mathring{W}_p^{1,2}(Q)$, $\lambda \in [0, 1]$, and $a \in \mathfrak{A}(1, \alpha)$. This fact should be considered well known (in case $p = 2$ it is found in [4]). It can be easily retrieved from Theorem 2.6 of [3] by following what is said in Remark 2.3 there. Even better way is to prove it directly as follows.

By an elementary Lemma 7 of [6], if $\delta \in (0, 1)$, then for any $a \in \mathfrak{A}(\delta, \delta^{-1})$, smooth $u(t, x)$ and $p > 1$

$$|u_t - u_{xx}|^p \leq (1 - \delta^2/2)^p (u_t^2 + u_{xx}^2)^{p/2} + (2/\delta)^p |u_t - au_{xx}|^p. \quad (2.4)$$

One also knows that

$$\|u\|_{1,2,p} := \left(\int_Q (u_t^2 + u_{xx}^2)^{p/2} dx dt \right)^{1/p}$$

defines an equivalent norm in $\mathring{W}_p^{1,2}(Q)$. Then observe that by integrating by parts or, better yet, using the Fourier transform we get that for any $u \in \mathring{W}_2^{1,2}(Q)$

$$\begin{aligned} \int_Q (u_t - u_{xx})^2 dx dt &= \|u\|_{1,2,2}^2 + 2 \int_Q u_t u_{xx} dx dt \\ &= \|u\|_{1,2,2}^2 + \int_{\mathbb{R}} |u_x(t, \cdot)|^2 dx \geq \|u\|_{1,2,2}^2. \end{aligned}$$

This implies that the norm of the inverse operator to $\partial_t - D^2 : \mathring{W}_2^{1,2}(Q) \rightarrow L_2(Q)$ is less than one. The Riesz's convexity theorem implies that the norm N_p of the inverse to $\partial_t - D^2 : \mathring{W}_p^{1,2}(Q) \rightarrow L_p(Q)$ is a continuous function of p and hence its product with $(1 - \delta^2/2)$ is less than ε , which is strictly less than one for p sufficiently close to 2 (with sufficiently

close defined by δ). Then owing to (2.4) we conclude that for any $u \in \overset{0}{W}_p^{1,2}(Q)$

$$\begin{aligned} \int_Q |u_t - u_{xx}|^p dx dt &\leq (1 - \delta^2/2)^p N_p^p \int_Q |u_t - u_{xx}|^p dx dt \\ &+ \frac{2^p}{\delta^p} \int_Q |u_t - au_{xx}|^p dx dt, \quad \int_Q |u_t - u_{xx}|^p dx dt \\ &\leq \frac{2^p}{\delta^p(1 - \varepsilon^p)} \int_Q |u_t - au_{xx}|^p dx dt, \\ \|u\|_{1,2,p} &\leq \frac{2^p N_p^p}{\delta^p(1 - \varepsilon^p)} \int_Q |u_t - au_{xx}|^p dx dt. \end{aligned}$$

The latter is an a priori estimate which allows one to prove the unique solvability for equation (1.1) for p close to 2 by the method of continuity.

The above arguments had, in particular, the goal to be combined with the maximum principle and provide for each $a \in \mathfrak{A}(\delta, \delta^{-1})$ a transition kernel $P_a(t, x, s, \Gamma)$ which is

- (i) a Borel function on $\{1 \geq t \geq s \geq 0\} \times \mathbb{R}$ for any Borel $\Gamma \subset \mathbb{R}$,
- (ii) a probability measure with respect to Γ whenever $1 \geq t \geq s \geq 0, x \in \mathbb{R}$,
- (iii) for any $\xi \in C_0^\infty(\mathbb{R})$ and $s \in [0, 1)$ the function

$$u(t, x) = \int_{\mathbb{R}} \xi(y) P_a(t, x, s, dy)$$

is a unique continuous in $[s, 1] \times \mathbb{R}$ solution of equation (1.1) belonging to $W_2^{1,2}((s, 1) \times \mathbb{R})$ and satisfying $u(s, x) = \xi(x)$. The details of construction of $P_a(t, x, s, \Gamma)$ can be found in [4].

Our next two results concern estimates on $P_a(t, x, s, \Gamma)$.

Theorem 2.9. *Let $\gamma \in (0, 1)$. Then there are constants $\nu \in (0, 1)$ and $\beta = \beta(\gamma) \in (1, \infty)$ depending only on γ such that for any $\varepsilon \in (0, 1/2)$*

$$\nu^{-1} \varepsilon^{\gamma-1} \geq \frac{1}{\varepsilon} \sup_{a \in \mathfrak{A}(1, \beta(\gamma))} P_a(1, 0, 0, [-\varepsilon, \varepsilon]) \geq \nu \frac{\varepsilon^{\gamma-1}}{|\ln \varepsilon|^{\gamma/2}}. \quad (2.5)$$

Furthermore, $\beta(\gamma) \rightarrow 1$ as $\gamma \uparrow 1$ and $\beta(\gamma) \rightarrow \infty$ as $\gamma \downarrow 0$.

Theorem 2.10. *Let $\gamma \in (1, 2)$. Then there are constants $\nu \in (0, 1)$ and $\beta = \beta(\gamma) \in (0, 1)$ depending only on γ such that for any $\varepsilon \in (0, 1/2)$*

$$\nu^{-1} \varepsilon^{\gamma-1} \geq \frac{1}{\varepsilon} \inf_{a \in \mathfrak{A}(\beta(\gamma), 1)} P_a(1, 0, 0, [-\varepsilon, \varepsilon]) \geq \nu \frac{\varepsilon^{\gamma-1}}{|\ln \varepsilon|^{\gamma/2}}. \quad (2.6)$$

Furthermore, $\beta(\gamma) \rightarrow 1$ as $\gamma \downarrow 1$ and $\beta(\gamma)$ tends to a nonzero limit as $\gamma \uparrow 2$.

3. AUXILIARY RESULTS

In this section we assume that we are given $\lambda \in \mathbb{R}$, $-\infty \leq a < b \leq \infty$ and a function $u > 0$ that is continuous on $[a, b] \cap \mathbb{R}$ and satisfies

$$u'' - 2xu' + 2\lambda u = 0 \quad (3.1)$$

on (a, b) .

Lemma 3.1. *The functions $u(-x)$ and*

$$w(x) = u(x) \int_a^x \frac{1}{u^2(t)} e^{t^2} dt$$

are solutions of (3.1) on $(-b, -a)$ and (a, b) , respectively.

Proof. Both assertions are consequences of direct calculations. For instance,

$$\begin{aligned} w' &= u' \int_a^x \frac{1}{u^2(t)} e^{t^2} dt + \frac{1}{u} e^{x^2}, \\ w'' &= u'' \int_a^x \frac{1}{u^2(t)} e^{t^2} dt + 2u' \frac{1}{u^2} e^{x^2} + u(-2u' \frac{1}{u^3} + 2x \frac{1}{u^2}) e^{x^2}, \end{aligned}$$

and the assertion about w follows.

Lemma 3.2. *The function u is a solution of (3.1) on (a, b) if and only if the function $w(x) = u(x)e^{-x^2}$ is a solution of*

$$w'' + 2xw' + 2\gamma w = 0 \quad (3.2)$$

on (a, b) , where $\gamma = \lambda + 1$.

The result follows from simple calculations showing that

$$\begin{aligned} u(x) &= w(x)e^{x^2}, \quad u'(x) = [w'(x) + 2xw(x)]e^{x^2}, \\ u''(x) &= [w''(x) + 4xw'(x) + (4x^2 + 2)w(x)]e^{x^2}. \end{aligned}$$

Lemma 3.3. *Let $w \not\equiv 0$ be a solution of (3.2) on \mathbb{R} with $\gamma > 0$ or $\gamma < -1$, and let $x_0 > 0$ be a point at which*

$$w'' = 0.$$

Then

- (i) $w'' > 0$ on any interval (x_0, a) , $a > x_0$, on which $w > 0$ and $w'' < 0$ on any interval (a, x_0) , $0 < a < x_0$, on which $w > 0$;
- (ii) $w'' < 0$ on any interval (x_0, a) , $a > x_0$, on which $w < 0$ and $w'' > 0$ on any interval (a, x_0) , $0 < a < x_0$, on which $w < 0$.

Proof. It suffices to prove (i). Observe that at any point $x > 0$ such that $w(x) > 0$ and $w''(x) = 0$ we have $xw' = -\gamma w$ and

$$w''' = -2w' - 2\gamma w' = -2(1 + \gamma)w' = 2\frac{\gamma(1 + \gamma)}{x}w > 0.$$

This easily implies our assertion and the lemma is proved.

Remark 3.4. Let w be any solution of (3.2), then $w''(x_0) = 0$ for $x_0 \neq 0$ may only hold if $w(x_0) \neq 0$, unless $w \equiv 0$. This follows from the uniqueness theorem for ODEs.

Lemma 3.5. *Assume that v and w are solutions of (3.2) on $[0, \infty)$ and $c_0 > 0$, $c_1 > 0$ are such that*

$$v(c_0) \neq 0, \quad w(c_1) \neq 0, \quad v''(c_0) = w''(c_1) = 0.$$

Define $a_1 = c_0/c_1$ and

$$\psi(x) = v^{-1}(c_0)v(x), \quad a(x) = 1 \quad \text{for } x \in [0, c_0],$$

$$\psi(x) = w^{-1}(c_1)w(x/a_1), \quad a(x) = a_1^2, \quad \text{for } x \in (c_0, \infty).$$

Then ψ is twice continuously differentiable on $[0, \infty)$, its second-order derivative is Lipschitz continuous in a neighborhood of c_0 , $a\psi''$ is continuously differentiable, $(a\psi'')$ is Lipschitz continuous in a neighborhood of c_0 , and ψ satisfies

$$a(x)\psi''(x) + 2x\psi'(x) + 2\gamma\psi(x) = 0 \tag{3.3}$$

on $[0, \infty)$.

Proof. By assumption (3.3) is satisfied on $[0, c_0)$. One easily checks that (3.3) is also satisfied on (c_0, ∞) . Since $\psi(c_0+) = \psi(c_0-) = 1$ and $\psi''(c_0+) = \psi''(c_0-) = 0$ we see that $\psi'(c_0+) = \psi'(c_0-)$, so that ψ is twice continuously differentiable. Furthermore, ψ'' has finite left and right derivatives at c_0 , so that it is Lipschitz continuous near this point. Our assertions concerning $a\psi''$ follow from the above and (3.3). The lemma is proved.

The following lemma shows the way we are going to use to construct our operators and functions while proving our main results.

Lemma 3.6. *Under the assumptions of Lemma 3.5 introduce*

$$\Psi(t, x) = \frac{1}{t^{\gamma/2}}\psi\left(\frac{|x|}{2\sqrt{t}}\right).$$

Then $\Psi \in C_{loc}^{1,2}((0, \infty) \times \mathbb{R})$ and Ψ satisfies

$$\Psi_t(t, x) = a(t, x)\Psi_{xx}(t, x),$$

where $a(t, x) = 1$ for $|x| \leq 2c_0\sqrt{t}$ and $a(t, x) = a_1^2$ for $|x| > 2c_0\sqrt{t}$.

This result follows from Lemma 3.5 and the fact that for $x \geq 0$

$$\begin{aligned}\Psi_t(t, x) &= \frac{1}{t^{1+\gamma/2}} \left(-\frac{\gamma}{2} \psi\left(\frac{x}{2\sqrt{t}}\right) - \frac{1}{2} \frac{x}{2\sqrt{t}} \psi'\left(\frac{x}{2\sqrt{t}}\right) \right) \\ &= \frac{a(t, x)}{4t^{1+\gamma/2}} \psi''\left(\frac{x}{2\sqrt{t}}\right),\end{aligned}$$

4. GENERAL PROPERTIES OF SOLUTIONS OF (3.1) AND (3.2) FOR $\gamma < 1, \lambda < 0$

Here we assume that $\gamma < 1$, so that $\lambda < 0$. One of solutions of (3.1) is

$$\phi(x) = \phi_\lambda(x) = \int_0^\infty e^{-2xr-r^2} r^{-1-\lambda} dr. \quad (4.1)$$

That ϕ is a solution of (3.1) follows from the fact that

$$\begin{aligned}\phi'(x) &= -2 \int_0^\infty e^{-2xr-r^2} r^{-\lambda} dr, \\ \phi''(x) &= 4 \int_0^\infty e^{-2xr-r^2} r^{1-\lambda} dr = -2 \int_0^\infty e^{-2xr} r^{-\lambda} de^{-r^2} \\ &= 2 \int_0^\infty e^{-2xr-r^2} [-2xr^{-\lambda} - \lambda r^{-1-\lambda}] dr.\end{aligned}$$

Observe that the change of variable $r = xs$ and sending $x \rightarrow \infty$ yields

$$\phi(x) \sim x^\lambda N_0, \quad \phi'(x) \sim x^{\lambda-1} N_1 \quad \phi''(x) \sim x^{\lambda-2} N_2, \quad (4.2)$$

where

$$\begin{aligned}N_0 &= N_{0\lambda} = \int_0^\infty e^{-2r} r^{-1-\lambda} dr, \\ N_1 &= N_{1\lambda} = -2 \int_0^\infty e^{-2r} r^{-\lambda} dr, \\ N_2 &= N_{2\lambda} = 4 \int_0^\infty e^{-2r} r^{1-\lambda} dr.\end{aligned}$$

To investigate the behavior of $\phi(x)e^{-x^2}$ as $x \rightarrow -\infty$ notice that $\phi(-x)e^{-x^2}, v(x) := [\phi(x) - \phi(-x)]e^{-x^2}$, and

$$w(x) := \phi(x)e^{-x^2} \int_0^x \frac{1}{\phi^2(t)} e^{t^2} dt$$

are solutions of (3.2). In addition

$$v(x)[2\phi'(0)]^{-1} = w(x)\phi(0) \quad (4.3)$$

due to the uniqueness theorem for ODEs. Then the formula

$$\phi(-x)e^{-x^2} = \phi(x)e^{-x^2} - v(x) \quad (4.4)$$

reduces the investigation of the behavior of $\phi(x)e^{-x^2}$ as $x \rightarrow -\infty$ to that of $w(x)$ and $\phi(x)e^{-x^2}$ as $x \rightarrow \infty$.

Lemma 4.1. *As $x \rightarrow \infty$,*

$$v(x) \sim \frac{\phi'(0)\phi(0)}{N_0}x^{-1-\lambda}, \quad \phi(-x) \sim \frac{\phi'(0)\phi(0)}{N_0}x^{-1-\lambda}e^{x^2}. \quad (4.5)$$

Proof. By (4.2) and l'Hospital's rule

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{w(x)}{x^{-1-\lambda}} &= N_0 \lim_{x \rightarrow \infty} \frac{\int_0^x \frac{1}{\phi^2(t)} e^{t^2} dt}{x^{-1-2\lambda} e^{x^2}} \\ &= N_0 \lim_{x \rightarrow \infty} \frac{1}{\phi^2(x)[-(1+2\lambda)x^{-2-2\lambda} + 2x^{-2\lambda}]} = \frac{1}{2N_0}, \end{aligned}$$

and the first relation in (4.5) holds due to (4.3). The second one follows from (4.4). The lemma is proved.

5. PROOF OF THEOREM 2.9

We split the proof of Theorem 2.9 into three parts: first we prove the estimate from above in (2.5), then the one from below, and finally we prove its last assertion. In this section as in Theorem 2.9, $\gamma \in (0, 1)$, $\lambda = \gamma - 1 \in (-1, 0)$. The function $\phi = \phi_\lambda$ is taken from (4.1).

Lemma 5.1. *Introduce*

$$u(x) = \phi(x) + \phi(-x).$$

Then there exists a unique $c_0 = c_{0\lambda} > 0$ such that $(u(x)e^{-x^2})'' = 0$ at $x = c_0$. In addition, $(u(x)e^{-x^2})'' < 0$ for $|x| < c_0$ and $(u(x)e^{-x^2})'' > 0$ for $|x| > c_0$. Furthermore, there exists a unique $c_1 = c_{1\lambda} > 0$ such that $(\phi(x)e^{-x^2})'' = 0$ at $x = c_1$. In addition, $(\phi(x)e^{-x^2})'' < 0$ for $0 < x < c_1$ and $(\phi(x)e^{-x^2})'' > 0$ for $x > c_1$.

Proof. Observe that at $x = 0$

$$(u(x)e^{-x^2})'' = 2\phi''(0) - 4\phi(0) = -4(1+\lambda)\phi(0) < 0.$$

In addition, according to (4.2) and (4.5), $u(x)e^{-x^2} \rightarrow 0$ as $x \rightarrow \infty$ and $u(0) > 0$. It follows that the graph of $u(x)e^{-x^2}$ has at least one inflection point on $(0, \infty)$. We denote by c_0 the smallest one. Since $u > 0$, Lemma 3.3 implies that $(u(x)e^{-x^2})'' < 0$ for $0 < x < c_0$, and $(u(x)e^{-x^2})'' > 0$ for $x > c_0$. In particular, $c_{0\lambda}$ is a unique positive solution of $(u(x)e^{-x^2})'' = 0$. By symmetry, $(u(x)e^{-x^2})'' < 0$ for $|x| < c_0$

and $(u(x)e^{-x^2})'' > 0$ for $|x| > c_0$. The same argument (apart from symmetry) works for $\phi(x)e^{-x^2}$ and the lemma is proved.

Introduce

$$\begin{aligned}
a_1 &= a_{1\gamma} = c_0/c_1 = c_{0\lambda}/c_{1\lambda}, \\
\hat{a}(x) &= \hat{a}_\gamma(x) = 1 \quad \text{for } |x| \leq c_0, \\
\hat{a}(x) &= \hat{a}_\gamma(x) = a_{1\gamma}^2 =: \beta(\gamma) \quad \text{for } |x| > c_0, \\
w(x) &= w_\gamma(x) = u^{-1}(c_0)u(x)e^{-x^2+c_0^2} \quad \text{for } |x| \leq c_0, \\
w(x) &= w_\gamma(x) = \phi^{-1}(c_1)\phi(|x|/a_1)e^{-(x/a_1)^2+c_1^2} \quad \text{for } |x| > c_0. \\
\Psi(t, x) &= \Psi^{(\gamma)}(t, x) = \frac{1}{t^{\gamma/2}}w(x/(2\sqrt{t})), \\
a^*(t, x) &= a_\gamma^*(t, x) = \hat{a}(x/(2\sqrt{t})).
\end{aligned} \tag{5.1}$$

Obviously,

$$a_\gamma^* \in \mathfrak{A}(1, \beta(\gamma)).$$

Remark 5.2. Lemma 3.5 implies that $w(x)$ has three bounded derivatives, which obviously tend to zero exponentially fast as $|x| \rightarrow \infty$. This yields that Ψ has three derivatives in (t, x) which are bounded in each set $\{t > \varepsilon\}$, where $\varepsilon > 0$ and tend to zero exponentially fast as $|x| \rightarrow \infty$ provided that t is restricted to a bounded interval separated from zero.

Lemma 3.5 also implies that $\hat{a}w''(x)$ has two bounded derivatives, which obviously tend to zero exponentially fast as $|x| \rightarrow \infty$. This yields that $a^*\Psi_{xx}$ has two derivatives in (t, x) which are bounded in each set $\{t > \varepsilon\}$, where $\varepsilon > 0$ and tend to zero exponentially fast as $|x| \rightarrow \infty$ provided that t is restricted to a bounded interval separated from zero.

Observe also that by Lemma 3.6

$$\Psi_t(t, x) = a^*(t, x)\Psi_{xx}(t, x) \tag{5.2}$$

and by Lemma 5.1

$$\Psi_{xx}(t, x) \leq 0 \quad \text{for } |x| \leq 2c_0\sqrt{t}, \quad \Psi_{xx}(t, x) \geq 0 \quad \text{for } |x| \geq 2c_0\sqrt{t}. \tag{5.3}$$

Lemma 5.3. *We have $c_1 < c_0$, $a_1 > 1$, and*

$$\Psi_t(t, x) = \max_{a \in [1, a_{1\gamma}^2]} [a\Psi_{xx}(t, x)]. \tag{5.4}$$

Proof. Assume that $c_1 \geq c_0$. Then $a_1 \leq 1$ and (5.2), (5.3) imply that

$$\begin{aligned}\Psi_t(t, x) &= \min_{a \in [a_1^2, 1]} [a \Psi_{xx}(t, x)], \\ \Psi_t(t, x) &\leq \Psi_{xx}(t, x).\end{aligned}$$

It follows by the maximum principle that

$$\begin{aligned}\frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} \Psi(t, y) e^{-(x-y)^2/4} dy &\geq \Psi(t+1, x), \\ \int_{\mathbb{R}} \Psi(t, y) dy &\geq 2\sqrt{\pi} \Psi(t+1, x).\end{aligned}$$

However, the integral on the left equals

$$2t^{(1-\gamma)/2} \int_{\mathbb{R}} w(y) dy \rightarrow 0$$

as $t \downarrow 0$. This yields a contradiction, hence $c_1 < c_0$, and the rest is trivial. The lemma is proved.

We now prove the estimate from above in (2.5). Recall that $\beta(\gamma) = a_{1\gamma}^2$ for $0 < \gamma < 1$.

Theorem 5.4. *For any $a \in \mathfrak{A}(a_{1\gamma}^2)$ and $\varepsilon \in (0, 1)$*

$$\frac{1}{\varepsilon} P_a(1, 0, 0, [-\varepsilon, \varepsilon]) \leq N \varepsilon^{\gamma-1},$$

where the constant N depends only on γ .

Proof. Since for any $t_0 > 0$ the function $\Psi(t_0 + t, x)$ satisfies

$$\Psi_t(t_0 + t, x) \geq a(t, x) \Psi_{xx}(t_0 + t, x),$$

by the maximum principle we have

$$\begin{aligned}\int_{\mathbb{R}} \Psi(t_0, y) P_a(1, x, 0, y) &\leq \Psi(t_0 + 1, x), \\ t_0^{-\gamma/2} \int_{\mathbb{R}} w(y/(2\sqrt{t_0})) P_a(1, 0, 0, y) &\leq \Psi(t_0 + 1, 0), \\ t_0^{-\gamma/2} \min_{|y| \leq 1/2} w(y) P_a(1, 0, 0, [-\sqrt{t_0}, \sqrt{t_0}]) &\leq \Psi(t_0 + 1, 0),\end{aligned}$$

and this proves the theorem.

Next, we prove the estimate from below in (2.5).

Theorem 5.5. *There is a constant $\nu > 0$ depending only on γ such that for any $\varepsilon \in (0, 1/2)$*

$$\frac{1}{\varepsilon} \sup_{a \in \mathfrak{A}(1, a_{1\gamma}^2)} P_a(1, 0, 0, [-\varepsilon, \varepsilon]) \geq \nu \frac{\varepsilon^{\gamma-1}}{|\ln \varepsilon|^{\gamma/2}}. \quad (5.5)$$

Proof. Fix a $t_0 \in (0, 1)$ and set $a(t, x) = a^*(t_0 + t, x)$, where a^* is introduced in (5.1). By comparing the equations satisfied by both sides of the following equation we come to a proof of the fact that

$$t_0^{-\gamma/2} \int_{\mathbb{R}} w(y/(2\sqrt{t_0})) P_a(t, x, 0, dy) = \Psi(t_0 + t, x). \quad (5.6)$$

It follows due to (4.2) that for any $c \geq c_0$ we have

$$t_0^{-\gamma/2} w(0) P_a(1, 0, 0, [-2c\sqrt{t_0}, 2c\sqrt{t_0}]) + N t_0^{-\gamma/2} c^\lambda e^{-(c/a_1)^2} \geq \Psi(t_0 + 1, 0),$$

where N is a constant depending only on γ . For

$$c = a_1 \sqrt{(\gamma/2) |\ln t_0|}$$

and t_0 small enough the second term on the left is less than $(1/2)\Psi(2, 0)$ due to the fact that $\lambda < 0$. Hence, with a constant $\alpha \geq 1$ depending only on γ for all small t_0

$$t_0^{-\gamma/2} w(0) P_a(1, 0, 0, [-\alpha\sqrt{t_0 |\ln t_0|}, \alpha\sqrt{t_0 |\ln t_0|}]) \geq (1/2)\Psi(2, 0). \quad (5.7)$$

Now denote

$$\varepsilon = \alpha\sqrt{t_0 |\ln t_0|}.$$

Then

$$t_0 = \frac{\varepsilon^2}{\alpha^2 |\ln t_0|}, \quad \ln \varepsilon = \ln \alpha + (1/2) \ln t_0 + (1/2) \ln |\ln t_0|,$$

and $|\ln t_0| \leq |\ln \varepsilon|$ for t_0 small enough, so that

$$t_0 \geq \frac{\varepsilon^2}{\alpha^2 |\ln \varepsilon|}.$$

This allows us to transform (5.7) into (5.5) for small ε , for which it only makes any real sense, and proves the theorem.

By comparing the behavior of $\varepsilon^{\gamma-1}$ for different γ and using the results of Theorems 5.4 and 5.5 we immediately come to the following.

Corollary 5.6. *The function $\beta(\gamma) = a_{1\gamma}^2$ is an increasing function of $\gamma \in (0, 1)$.*

Finally we deal with the last assertion of Theorem 2.9. It suffices to prove that

$$\lim_{\gamma \uparrow 1} c_{1\gamma} = 2^{-1/2}, \quad \lim_{\gamma \uparrow 1} c_{0\gamma} = 2^{-1/2}, \quad (5.8)$$

$$\lim_{\gamma \downarrow 0} c_{0\gamma}/c_{1\gamma} = \infty. \quad (5.9)$$

Observe that if a function $v > 0$ satisfies (3.1), then the inequality $(ve^{-x^2})'' > 0$ is written as

$$v'' - 4xv' + 2v(2x^2 - 1) > 0, \quad -2xv' + 2v(2x^2 - 1 - \lambda) > 0,$$

$$\frac{xv'(x)}{v(x)} < 2x^2 - 1 - \lambda. \quad (5.10)$$

Accordingly, since ϕ satisfies (3.1), equation $(\phi e^{-x^2})'' = 0$, defining a unique $c_1 = c_{1\gamma} > 0$, transforms into

$$\frac{c_1\phi'(c_1)}{\phi(c_1)} = 2c_1^2 - 1 - \lambda. \quad (5.11)$$

Since $\phi' < 0$ we conclude that the right-hand side of (5.11) is negative and $c_{1\gamma}$ is a bounded function of $\gamma \in (0, 1)$. Furthermore, obviously ϕ' is bounded on any interval $[0, b]$ by a constant independent of $\gamma \in [1/2, 1)$ and $\phi \rightarrow \infty$ uniformly on any such interval as $\gamma \uparrow 1$, so that $\lambda \uparrow 0$. Now the first relation in (5.8) follows from (5.11).

To prove the second one, observe that $c_0 = c_{0\gamma}$ is defined as a unique positive solution of

$$c_0 \frac{\phi'(c_0) - \phi'(-c_0)}{\phi(c_0) + \phi(-c_0)} = 2c_0^2 - 1 - \lambda \quad (5.12)$$

and

$$c_0\phi'(c_0) - c_0\phi'(-c_0) = 4c_0 \int_0^\infty e^{-r^2} r^{-\lambda} \sinh(2c_0 r) dr,$$

$$\phi(c_0) + \phi(-c_0) = 2 \int_0^\infty e^{-r^2} r^{-1-\lambda} \cosh(2c_0 r) dr.$$

Furthermore, from the above argument concerning (5.10) and Lemma 5.1 we know that if $x > 0$ and

$$x \frac{\phi'(x) - \phi'(-x)}{\phi(x) + \phi(-x)} < 2x^2 - 1 - \lambda,$$

then $c_0 \leq x$. For $x = 1$ the left-hand side obviously tends to zero as $\gamma \uparrow 1$ ($\lambda \uparrow 0$). This yields the boundedness of $c_{0\gamma}$ and we can finish the proof of the second relation in (5.8) as before.

To prove (5.9) we look at (5.11) as quadratic equation relative to c_1 and then find that

$$c_1 = 2(1 + \lambda)/B,$$

where (recall that $\phi' < 0$)

$$B = B_\lambda = |\phi'(c_1)|\phi^{-1}(c_1) + \sqrt{|\phi'(c_1)|^2\phi^{-2}(c_1) + 8(1 + \lambda)}.$$

Since $|\phi'(x)|\phi^{-1}(x) \not\rightarrow 0$ as $x \downarrow 0$, we conclude that

$$c_{1\gamma} = O(\gamma)$$

as $\gamma \downarrow 0$.

Now if we assume that along a sequence $\gamma_n \downarrow 0$ we have $c_{0\gamma_n}/c_{1\gamma_n} \rightarrow \beta < \infty$, then $c_{0\gamma_n} \rightarrow 0$, and after dividing both parts of (5.12) by $c_{0\gamma}^2$ and passing to the limit along the subsequence we get that

$$\frac{\phi_{-1}''(0)}{\phi_{-1}(0)} = 2 - \lim_{n \rightarrow \infty} \frac{\gamma_n}{c_{0\gamma_n}^2}. \quad (5.13)$$

The function ϕ_{-1} satisfies (3.1) with $\lambda = -1$, therefore, $\phi_{-1}''(0) = 2\phi_{-1}(0)$, and (5.13) implies that

$$\lim_{n \rightarrow \infty} \frac{\gamma_n}{c_{0\gamma_n}^2} = 0, \quad \lim_{n \rightarrow \infty} \frac{c_{1\gamma_n}}{c_{0\gamma_n}^2} = 0, \quad \frac{1}{\beta^2} \lim_{n \rightarrow \infty} \frac{1}{c_{1\gamma_n}} = 0,$$

and the latter is impossible. This proves (5.9) and brings the proof of Theorem 2.9 to an end.

6. PROOF OF THEOREMS 2.1 AND 2.3

For $p \in (1, 3/2)$ one can define a function $\gamma(p) \in (0, 1)$ so that

$$\gamma(p) < \frac{3}{p} - 2, \quad \lim_{p \downarrow 1} \gamma(p) = 1.$$

We take any such function $\gamma(p)$ and set

$$\alpha(p) = \beta(\gamma(p)) = a_{1\gamma(p)}^2.$$

Obviously,

$$a_{\gamma(p)}^* \in \mathfrak{A}(1, \alpha(p)).$$

Theorem 2.1 is a consequence of the following.

Theorem 6.1. *Let $p \in (1, 3/2)$. Then the equation*

$$u_t = a_{\gamma(p)}^* u_{xx}$$

has a nonzero unbounded solution of class $\overset{0}{W}_p^{1,2}(Q)$. Furthermore, $\alpha(p) \rightarrow 1$ as $p \downarrow 1$ and $\alpha(p) \rightarrow \infty$ as $p \uparrow 3/2$

Proof. Simple computations show that if we extend $\Psi^{(\gamma(p))}(t, x)$ as zero for $t \leq 0$, then the unbounded function we obtain will belong to $W_p^{1,2}((-1, 1) \times \mathbb{R})$ and, to prove the first assertion, it only remains to recall that $\Psi^{(\gamma)}$ satisfies (5.2). The second assertion follows immediately by the construction of $\gamma(p)$ and Theorem 2.9. The theorem is proved.

Next argument is based on a general rule that if a linear homogeneous equation has a nonzero solution u , then the adjoint equation is only solvable if its right-hand side is orthogonal to u . We apply this rule to $u = \Psi_{xx}$.

Here is a result implying Theorem 2.3.

Theorem 6.2. *Let $p \in (3, \infty)$. Then for $q = p/(p-1)$ and $\alpha = a_{1\gamma(q)}^2$ there exists a function $a \in \mathfrak{A}(\alpha)$, and an $f \in L_p(Q)$ such that the equation*

$$u_t = au_{xx} + f \quad (6.1)$$

has no solutions of class $\overset{0}{W}_p^{1,2}(Q)$.

Proof. Define

$$\Phi(t, x) = \Psi^{(\gamma(q))}(1-t, x), \quad a(t, x) = a_{\gamma(q)}^*(1-t, x),$$

and take any $f \in L_p(Q)$ such that $\Phi f \in L_1(Q)$ and

$$\int_Q \Phi_{xx} f \, dx dt \neq 0.$$

Assume that u is a solution of (6.1) of class $\overset{0}{W}_p^{1,2}(Q)$. We multiply (6.1) through by Φ_{xx} and use that $a\Phi_{xx} = -\Phi_t$. Then we have the following

$$\begin{aligned} \int_0^{t_0} \int_{\mathbb{R}} \Phi_{xx} u_t \, dx dt &= - \int_0^{t_0} \int_{\mathbb{R}} \Phi_t u_{xx} \, dx dt + \int_0^{t_0} \int_{\mathbb{R}} \Phi_{xx} f \, dx dt, \\ \int_0^{t_0} \int_{\mathbb{R}} \frac{\partial}{\partial t} [\Phi_{xx} u] \, dx dt &= \int_0^{t_0} \int_{\mathbb{R}} \Phi_{xx} f \, dx dt =: \kappa(t_0), \\ \int_{\mathbb{R}} u_x(t_0, x) \Psi_x(1-t_0, x) \, dx &= \kappa(t_0), \end{aligned} \quad (6.2)$$

where the last equality is obtained by integrating by parts and using Remark 5.2.

Next, we are going to use an embedding theorem that is Lemma 2.3.3 of [10] according to which for each $t \in [0, 1]$, $u(t, \cdot) \in C^{1+\varepsilon}(\mathbb{R})$ and the norm of $u(t, \cdot)$ in this space is a bounded function of t . Here $\varepsilon > 0$ is any number such that

$$1 - \frac{3}{p} > \varepsilon > \gamma(q).$$

That such an ε exists follows from the fact that the inequalities $1 - 3/p > \gamma(q)$ and $\gamma(q) < 3/q - 2$ are equivalent.

Since $\Psi_x(1-t_0, x)$ is an odd function of x we can replace $u_x(t_0, x)$ in (6.2) with $u_x(t_0, x) - u_x(t_0, 0)$, and since the latter by magnitude is less than $N|x|^\varepsilon$, where N is a constant, we come to the conclusion that

$$|\kappa(t_0)| \leq \frac{N}{(1-t_0)^{(1+\gamma)/2}} \int_{\mathbb{R}} |x|^\varepsilon |w'(x/(2\sqrt{1-t_0}))| \, dx,$$

where $\gamma = \gamma(q)$.

However, the change of variable $y = x/(2\sqrt{1-t_0})$ shows that the expression on the right equals a constant times $(1-t_0)^{(\varepsilon-\gamma)/2}$, which tends to zero as $t_0 \uparrow 1$. Hence, $\kappa(t_0) \rightarrow 0$ and this is the desired contradiction proving the theorem.

7. PROOF OF THEOREMS 2.5 AND 2.6

To prove Theorem 2.5 it suffices to take

$$a = a_{\gamma(p)}^*, \quad u = \Psi_x^{(\gamma(p))}$$

and use what is said in the proof of Theorem 6.1 and also recall the definition of $\mathcal{H}_p^1(Q)$ and use Remark 5.2.

While proving Theorem 2.6 we define Φ and a as in the proof of Theorem 6.2 and take any $f \in L_p(Q)$ such that $f\Lambda\Phi_x \in L_1(Q)$ and

$$\int_Q f\Lambda\Phi_x dxdt \neq 0$$

and let u be a solution of (2.3) of class $\mathcal{H}_p^{1,2}(Q)$.

Then observe that $\Phi_x(t, \cdot)$ is a strongly continuous differentiable $H_q^1(\mathbb{R})$ -valued function on $[0, 1)$ for any $q \in (1, \infty)$, in particular, for $q = p/(p-1)$. Since $u(t, \cdot)$ is a strongly differentiable $H_p^{-1}(\mathbb{R})$ -valued function on $[0, 1]$ we can write

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} u(t, x) \Phi_x(t, x) dx &= \frac{d}{dt} \langle u(t, \cdot), \Phi_x(t, \cdot) \rangle = \langle u_t(t, \cdot), \Phi_x(t, \cdot) \rangle \\ &\quad + \langle u(t, \cdot), \Phi_{tx}(t, \cdot) \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $H_q^1(\mathbb{R})$ and $H_p^{-1}(\mathbb{R})$. Next we use the fact that $\Phi_{tx}(t, \cdot)$ and $u_x(t, \cdot)$ are usual functions (for almost all t), so that

$$\langle u(t, \cdot), \Phi_{tx}(t, \cdot) \rangle = \int_{\mathbb{R}} u(t, x) \Phi_{tx}(t, x) dx = - \int_{\mathbb{R}} u_x(t, x) \Phi_t(t, x) dx.$$

We also note that (for almost all t)

$$\langle u_t(t, \cdot), \Phi_x(t, \cdot) \rangle = \int_{\mathbb{R}} f(t, x) \Lambda \Phi_x(t, x) dx + \langle (au_x(t, \cdot))_x, \Phi_x(t, \cdot) \rangle,$$

where the last term equals

$$-\langle u_x(t, \cdot), a\Phi_{xx}(t, \cdot) \rangle = \int_{\mathbb{R}} u_x(t, x) \Phi_t(t, x) dx.$$

By combining these arguments we get that

$$\frac{d}{dt} \int_{\mathbb{R}} u(t, x) \Phi_x(t, x) dx = \int_{\mathbb{R}} f(t, x) \Lambda \Phi_x(t, x) dx$$

and for any $t_0 \in [0, 1)$

$$\int_{\mathbb{R}} u(t_0, x) \Phi_x(t_0, x) dx = \int_0^{t_0} \int_{\mathbb{R}} f \Lambda \Phi_x dx dt.$$

Finally, recall that $u = \Lambda w$ where $w \in \overset{0}{W}_p^{1,2}(Q)$, which by Lemma 2.3.3 of [10] implies that u is Hölder continuous in x with the same exponent as in the proof of Theorem 6.1 and this allows us to get a contradiction by letting $t_0 \uparrow 1$ in the same way as in that proof. Both theorems are thus proved.

8. PROOF OF THEOREM 2.10

Here we suppose that $\gamma \in (1, 2)$, so that $\lambda \in (0, 1)$. One of solutions of (3.1) is

$$\phi(x) = \phi_\lambda(x) = \int_0^\infty [1 - e^{-2xr-r^2}] \frac{1}{r^{1+\lambda}} dr,$$

with

$$\begin{aligned} \phi'(x) &= 2 \int_0^\infty e^{-2xr-r^2} \frac{1}{r^\lambda} dr > 0, \\ \phi''(x) &= -4 \int_0^\infty e^{-2xr-r^2} r^{1-\lambda} dr < 0. \end{aligned}$$

The fact that it is indeed a solution is seen from the following:

$$\begin{aligned} \phi(x) &= -\frac{1}{\lambda} \int_0^\infty [1 - e^{-2xr-r^2}] dr^{-\lambda} \\ &= \frac{1}{\lambda} \int_0^\infty (2x + 2r) e^{-2xr-r^2} r^{-\lambda} dr. \end{aligned} \quad (8.1)$$

Observe that, as is easily seen (after substituting $r = xs$), as $x \rightarrow \infty$

$$\phi(x) \sim x^\lambda \int_0^\infty [1 - e^{-2r}] \frac{1}{r^{1+\lambda}} dr, \quad \phi'(x) \sim 2x^{\lambda-1} \int_0^\infty e^{-2r} \frac{1}{r^\lambda} dr. \quad (8.2)$$

Obviously $\phi(x) \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$ and $\phi(0) > 0$. Therefore we can define $x_0 = x_{0\lambda} < 0$ as a unique root of

$$\phi(x_0) = 0$$

and for $c > x_0$ let

$$\psi_c(x) = \phi(x) \int_c^x \frac{1}{\phi^2(t)} e^{t^2} dt$$

As in Section 4 we use this function to investigate the behavior of $\phi(-x)$ as $x \rightarrow \infty$.

By Lemma 3.1 the function $\psi_c(x)$ satisfies (3.1) for any $c > c_0$. Also both functions ψ_{-c_0} and $\phi(-x)$ vanish at $x = c_0$ and hence there is a constant $m > 0$ such that

$$\phi(-x) = -m\psi_{-c_0}(x).$$

In the same way as Lemma 4.1 is proved one gets the following.

Lemma 8.1. *We have*

$$\psi_c(x) \sim N \frac{1}{x^{1+\lambda}} e^{x^2}, \quad \phi(-x) \sim -N \frac{1}{x^{1+\lambda}} e^{x^2}$$

as $x \rightarrow \infty$, where the constants $N > 0$ depend only on λ .

Next we set

$$u(x) = \phi(x) + \phi(-x)$$

and in the same way as in Lemma 5.1 we prove that there exists a unique $c_1 = c_{1\lambda} > 0$ such that $(\phi(x)e^{-x^2})'' = 0$ at $x = c_1$. In addition,

$$(\phi(x)e^{-x^2})'' < 0 \quad \text{for } 0 < x < c_1, \quad (\phi(x)e^{-x^2})'' > 0 \quad \text{for } x > c_1. \quad (8.3)$$

Furthermore as in Lemma 5.1, the graph of the function $u(x)e^{-x^2}$ has inflection points on $(0, \infty)$ and we denote by $c_{0\lambda}$ the smallest one. Observe that

$$u(c_{0\lambda}) > 0. \quad (8.4)$$

Indeed, the equality $u(c_{0\lambda}) = 0$ is impossible due to Remark 3.4. However, if $u(c_{0\lambda}) < 0$, then by Lemma 3.3 (ii) and Remark 3.4 the second-order derivative of $u(x)e^{-x^2}$ at the closest zero of $u(x)$ lying to the left of $c_{0\lambda}$ is strictly positive and being negative at the origin it would have another root smaller than $c_{0\lambda}$, which contradicts its definition. Thus,

$$u(c_{0\lambda}) > 0, \quad (u(x)e^{-x^2})'' < 0 \quad \text{for } 0 < x < c_{0\lambda}. \quad (8.5)$$

By the way, notice that, according to (8.2) and Lemma 8.1 the function $u_\lambda(x)e^{-x^2}$ approaches zero from the negative side as $x \rightarrow \infty$. Therefore, there exists the smallest root $y > 0$ of the equation $u_\lambda(x) = 0$ and as follows from the above $y > c_{0\lambda}$ and there are no inflection points between $c_{0\lambda}$ and y . Then in the same way in which (4.3) is obtained one shows that for $x \geq y$

$$u_\lambda(x) = -\kappa\phi_\lambda(x) \int_y^x \frac{1}{\phi_\lambda^2(t)} e^{t^2} dt,$$

where $\kappa \in (0, \infty)$ is a constant. It follows that $u(x) < 0$ for all $x > y$. Then the existence of a unique root of the equation $(u_\lambda(x)e^{-x^2})'' = 0$ lying on (y, ∞) is obtained as above, so that $c_{0\lambda}$ is the smallest of the two roots.

Then we introduce $a_1 = a_{1\gamma}$, $\beta(\gamma) = a_{1\gamma}^2$, $w = w_\gamma$, $\Psi = \Psi^{(\gamma)}$ and $a^* = a_\gamma^*$ in the same way as in (5.1).

As before, (5.2) and (5.3) hold. However, in contrast with Lemma 5.3 we now have the following.

Lemma 8.2. *We have $c_1 > c_0$, $a_1 < 1$, and*

$$\Psi_t(t, x) = \min_{a \in [a_1^2, 1]} [a \Psi_{xx}(t, x)]. \quad (8.6)$$

Proof. Assume that $c_0 \geq c_1$. Then $a_1 \geq 1$ and (5.2), (5.3) imply that

$$\begin{aligned} \Psi_t(t, x) &= \max_{a \in [1, a_1^2]} [a \Psi_{xx}(t, x)], \\ \Psi_t(t, x) &\geq \Psi_{xx}(t, x). \end{aligned}$$

It follows by the maximum principle that

$$\begin{aligned} \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} \Psi(t, y) e^{-(x-y)^2/4} dy &\leq \Psi(t+1, x), \\ 2t^{-\gamma/2} \int_{\mathbb{R}} w(y/(2\sqrt{t})) e^{-y^2/4} dy &\leq 2\sqrt{\pi} \Psi(t+1, 0). \end{aligned}$$

However, the integral on the left is equivalent to

$$4t^{(1-\gamma)/2} \int_{\mathbb{R}} w(y) dy \rightarrow \infty$$

as $t \downarrow 0$. This yields a contradiction, hence $c_1 > c_0$, and the rest is trivial. The lemma is proved.

We are now ready to prove part of Theorem 2.10.

Theorem 8.3. *There exists a constant $\nu \in (0, 1)$ depending only on $\gamma \in (1, 2)$ such that for any $\varepsilon \in (0, 1/2)$*

$$\frac{1}{\varepsilon} \inf_{a \in \mathfrak{A}(a_{1\gamma}^2, 1)} P_a(1, 0, 0, [-\varepsilon, \varepsilon]) \geq \nu \frac{\varepsilon^{\gamma-1}}{|\ln \varepsilon|^{\gamma/2}}. \quad (8.7)$$

Proof. Take an $a \in \mathfrak{A}(a_{1\gamma}^2, 1)$, Since

$$\Psi_t \leq a \Psi_{xx},$$

we have

$$\begin{aligned} \int_{\mathbb{R}} \Psi(t, y) P_a(1, x, 0, dy) &\geq \Psi(1+t, x), \\ \int_{\mathbb{R}} \Psi(t, y) P_a(1, 0, 0, dy) &\geq \Psi(1+t, 0) \geq \Psi(2, 0), \end{aligned}$$

where the last inequality holds for $t \in (0, 1]$. Here

$$\int_{\mathbb{R}} \Psi(t, y) P_a(1, 0, 0, dy) \leq Mt^{-\gamma/2} P_a(1, 0, 0, [-2\sqrt{t|\ln t|}, 2\sqrt{t|\ln t|}])$$

$$+t^{-\gamma/2}w(\sqrt{|\ln t|}),$$

where $M = \max w$. As $t \downarrow 0$ we have

$$t^{-\gamma/2}w(\sqrt{|\ln t|}) \leq Nt^{-\gamma/2}|\ln t|^{\lambda/2}e^{-|\ln t|} = Nt^{1-\gamma/2}|\ln t|^{\lambda/2} \rightarrow 0.$$

It follows that there exists $t_0 \in (0, 1/2)$ such that

$$t^{-\gamma/2}w(\sqrt{|\ln t|}) \leq (1/2)\Psi(2, 0)$$

for $t \in (0, t_0]$. In that case upon setting $\varepsilon = 2\sqrt{t|\ln t|}$ we obtain

$$\begin{aligned} \frac{1}{\varepsilon}P_a(1, 0, 0, [-\varepsilon, \varepsilon]) &\geq M^{-1}\Psi(2, 0)t^{(\gamma-1)/2}\frac{1}{2\sqrt{|\ln t|}} \\ &= 2^{-1}M^{-1}\Psi(2, 0)\varepsilon^{\gamma-1}\frac{1}{|\ln t|^{\gamma/2}}. \end{aligned}$$

To get (8.7), now it only remains to observe that

$$|\ln t| = 2\ln 2 + 2|\ln \varepsilon| + \ln |\ln t| \leq 2|\ln \varepsilon| + (1/2)|\ln t|$$

for $t \in (0, t_1]$ with sufficiently small $t_1 > 0$, so that

$$\frac{1}{|\ln t|^{\gamma/2}} \geq \delta \frac{1}{|\ln \varepsilon|^{\gamma/2}}.$$

The theorem is proved.

Now we are concerned with another part of Theorem 2.10.

Theorem 8.4. *For $\varepsilon \in (0, 1)$ set $a^{(\varepsilon)}(t, x) = a^*(\varepsilon + t, x)$. Then we have*

$$\frac{1}{\varepsilon}P_{a^{(\varepsilon)}}(1, 0, 0, [-\varepsilon, \varepsilon]) \leq \mu^{-1}\Psi(1, 0)\varepsilon^{\gamma-1},$$

where $\mu = \inf_{|x| \leq 1/2} w(x)$ depends only on γ .

Proof. Indeed, we have

$$\int_{\mathbb{R}} \Psi(t, y) P_{a^{(\varepsilon)}}(1, 0, 0, dy) = \Psi(1+t, 0) \leq \Psi(1, 0),$$

$$t^{-\gamma/2}\mu P_{a^{(\varepsilon)}}(1, 0, 0, [-\sqrt{t}, \sqrt{t}]) \leq \Psi(1, 0).$$

This obviously leads to the desired result and proves the theorem.

Similarly to Corollary 5.6 we have

Corollary 8.5. *The function $\beta(\gamma) = a_{1\gamma}^2$ is a decreasing function of $\gamma \in (1, 2)$.*

To finish the proof of Theorem 2.10 we only need to show that $\beta(\gamma) = a_{1\gamma}^2 \rightarrow 1$ as $\gamma \downarrow 1$ and $\beta(\gamma)$ tends to a nonzero limit as $\gamma \uparrow 2$. We already know that these limits exist.

Observe that the condition $(ve^{-x^2})'' = 0$ for solutions $v > 0$ of (3.1) is also written as

$$\frac{v''(x)}{2v(x)} = 2x^2 - 1 - 2\lambda.$$

Therefore, $c_0 = c_{0\gamma}$ and $c_1 = c_{1\gamma}$ also satisfy

$$\frac{\phi_\lambda''(c_0) + \phi_\lambda''(-c_0)}{2\phi_\lambda(c_0) + 2\phi_\lambda(-c_0)} = 2c_0^2 - 1 - 2\lambda, \quad \frac{\phi_\lambda''(c_1)}{2\phi_\lambda(c_1)} = 2c_1^2 - 1 - 2\lambda. \quad (8.8)$$

Here $\phi_\lambda'' \leq 0$ and $\phi_\lambda(c_0) + \phi_\lambda(-c_0) > 0$ owing to (8.4). Hence $c_{0\gamma}$ and $c_{1\gamma}$ are bounded. Also ϕ_λ'' is bounded on bounded intervals and $\phi_\lambda \rightarrow \infty$ uniformly on intervals of type $[0, b]$, $b > 0$, as $\gamma = \lambda + 1 \downarrow 1$ because of the divergence at infinity of the integral defining ϕ_λ . This immediately implies that

$$\lim_{\gamma \downarrow 1} c_{1\gamma} = 2^{-1/2} \quad (8.9)$$

and, along with Corollary 8.5, shows that the limit of $c_{0\gamma}$ as $\gamma \downarrow 1$ exists. We denote it by c . By using (8.1) we rewrite the first equation in (8.8) as

$$\begin{aligned} & \lambda[\phi_\lambda''(c_0) + \phi_\lambda''(-c_0)] \\ &= 2[2c_0^2 - 1 - 2\lambda] \int_0^\infty [(2c_0 + 2r)e^{-2c_0r} + (2r - c_0)e^{2c_0r}]e^{-r^2}r^{-\lambda} dr. \end{aligned}$$

By letting $\lambda \downarrow 0$ we obtain

$$2[2c^2 - 1] \int_0^\infty [(2c + 2r)e^{-2cr} + (2r - c)e^{2cr}]e^{-r^2} dr = 0. \quad (8.10)$$

As a function, say f , of c the last integral is obtained as the limit as $\lambda \downarrow 0$ of solutions of the equation $v'' - 2xv' + \lambda v = 0$. Therefore, $f'' - 2xf' = 0$,

$$f' = C_1 e^{x^2}, \quad f(x) = C_1 \int_0^x e^{t^2} dt + C_2,$$

where C_1, C_2 are some constants. The function f is obviously even, hence $C_1 = 0$. Also $C_2 = f(0) > 0$ and (8.10) implies that

$$c = \lim_{\gamma \downarrow 1} c_{0\gamma} = 2^{-1/2}, \quad \lim_{\gamma \downarrow 1} \beta(\gamma) = 1.$$

Next, by comparing (8.8) with (5.11), the left-hand sides of which is positive in our situation, we obtain

$$0 \leq 2c_1^2 - 1 - \lambda \leq \lambda, \quad 1 + \lambda < 2c_1^2 < 1 + 2\lambda$$

which implies (8.9) one more time and also shows that $c_{1\gamma}$ is separated away from zero. On the set of possible values of $c_{1\gamma}$ we have that $\phi_\lambda \rightarrow \infty$ because of the divergence at zero of the above mentioned integral as $\lambda \uparrow 1$. Hence, by (8.8)

$$\lim_{\gamma \uparrow 2} c_{1\gamma} = (3/2)^{1/2}.$$

Again this implies that the limit of $c_{0\gamma}$ as $\gamma \uparrow 2$ exists. Moreover, as is easy to see, $\phi_\lambda(x) + \phi_\lambda(-x)$ tends to a finite limit, say ψ , as $\lambda \uparrow 1$ and therefore $c_{0\gamma}$ tends to, say, $d \geq 0$ satisfying

$$\psi''(d) = 2\psi(d)(2d^2 - 3).$$

Since $\phi_\lambda(x) + \phi_\lambda(-x)$ are solutions of (3.1) we have that $\psi'' - 2x\psi' + 2\psi = 0$, which shows that $\psi''(0) = -2\psi(0)$. In particular, $d \neq 0$ and this finishes the proof of the theorem.

REFERENCES

- [1] Hongjie Dong and Doyoon Kim, *On the impossibility of W_p^2 estimates for elliptic equations with piecewise constant coefficients*, J. Funct. Anal., Vol. 267 (2014), No. 10, 3963–3974.
- [2] Hongjie Dong and Doyoon Kim, *Elliptic and parabolic equations with measurable coefficients in weighted Sobolev spaces*, Adv. Math., Vol. 274 (2015), 681–735.
- [3] Hongjie Dong and N.V. Krylov, *Second-order elliptic and parabolic equations with $B(\mathbb{R}^2, VMO)$ coefficients*, Trans. Amer. Math. Soc., Vol. 362 (2010), No. 12, 6477–6494.
- [4] E.B. Fabes and C.E. Kenig, *Examples of singular parabolic measures and singular transition probability densities*, Duke Math. J., Vol. 48 (1981), No. 4, 845–856.
- [5] A.M. Il'in, *On the fundamental solution of a parabolic equation*, Dokl. Akad. Nauk SSSR, Vol. 147 (1962), 768–771 in Russian. English translation in Soviet Math. Dokl., Vol. 3 (1962) 1697–1700.
- [6] N. V. Krylov, *On equations of minimax type in the theory of elliptic and parabolic equations in the plane*, Matematicheski Sbornik, Vol. 81 (1970), No. 1, 3–22 in Russian; English translation in Math. USSR Sbornik, Vol. 10 (1970), 1–20.
- [7] *Brownian trajectory is a regular lateral boundary for the heat equation*, SIAM J. Math. Anal., Vol. 34, No. 5 (2003), 1167–1182.
- [8] N.V. Krylov, *About an example of N.N. Ural'tseva and weak uniqueness for elliptic operators*, pp. 131–144 in AMS Transl., Ser. 2, Advances in the Mathematical Sciences, Nonlinear Partial Differential Equations and Related Topics: Dedicated to Nina N. Uraltseva; A.A. Arkhipova, A.I. Nazarov eds, Vol. 229, 2010.
- [9] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Линейные и квазилинейные уравнения эллиптического типа* (Russian) [Linear and quasilinear equations of elliptic type] Second edition, revised. Izdat. "Nauka", Moscow, 1973. 576 pp.

- [10] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'tseva, "Linear and quasi-linear parabolic equations", Nauka, Moscow, 1967, in Russian; English translation: Amer. Math. Soc., Providence, RI, 1968, http://bookfi.org/fs/101/52932_aa51ff
- [11] E. Perkins, *On the Hausdorff dimension of the Brownian slow points*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, Vol. 64 (1983), 369-399.
- [12] M.V. Safonov *An example of diffusion process with singular distribution at some given time*, Abstr. Comm. Third Vilnius Conference on Probability Theory and Math. Statistics, June 22-27, 1981, Vilnius, 133-134 (Russian).

127 VINCENT HALL, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN, 55455
E-mail address: krylov@math.umn.edu